

ORIGINAL ARTICLE

A NOTE ON EFFICIENT FITTING OF STOCHASTIC VOLATILITY MODELS

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The stochastic volatility model is a popular tool for modeling the volatility of assets. The model is a nonlinear and non-Gaussian state space model and presents some challenges not seen in general. Many approaches have been developed for Bayesian analysis that rely on numerically intensive techniques such as Markov chain Monte Carlo (MCMC). Convergence and mixing problems still plague MCMC algorithms used for the model. We present an approach that ameliorates the slow convergence and mixing problems when fitting stochastic volatility models. The approach accelerates the convergence by exploiting the geometry of one of the targets. We demonstrate the method on various numerical examples.

Received 24 July 2019; Accepted 02 September 2020

Keywords: Ancestral sampling; efficient Markov chain Monte Carlo; particle Gibbs; stochastic volatility.

1. THE PROBLEM

Most models of volatility that are used in practice are of a multiplicative form, modeling the return of an asset, say y_t , observed at discrete time points, $t = 1, \dots, n$, as

$$y_t = \sigma_t \epsilon_t, \quad (1)$$

where ϵ_t is an iid sequence and the volatility process σ_t is a non-negative stochastic process such that ϵ_t is independent of σ_s for all $s \leq t$. It is often assumed that ϵ_t has zero mean and unit variance.

The basic univariate discrete-time stochastic volatility (SV) model writes the returns and the non-anticipative log-volatility process, $x_t = \log \sigma_t^2$, as

$$x_t = \mu + \phi(x_{t-1} - \mu) + \sigma w_t, \quad (2)$$

$$y_t = \beta \exp\left\{\frac{1}{2}x_t\right\}\epsilon_t, \quad (3)$$

where $x_0 \sim N(\mu, \frac{\sigma^2}{1-\phi^2})$, $w_t \stackrel{\text{iid}}{\sim} N(0, 1)$, and $\epsilon_t \stackrel{\text{iid}}{\sim} N(0, 1)$ are all independent processes. The volatility process x_t is not observed directly, but only through the observations, y_t . The detailed econometric properties of the model are discussed in Shephard (1996) and Taylor (1994, 2008).

The model (2)–(3) is a nonlinear state space model and Bayesian analysis of such models can be approached using various methods, many of which are described in Douc *et al.* (2014, Chap. 12). Early MCMC approaches to the problem may be found in Carlin *et al.* (1992), Kim *et al.* (1998), Jacquier *et al.* (1994), and Taylor (1994). Other more current approaches may be found in Kastner and Frühwirth-Schnatter (2014), with techniques that are implemented in the R package `stochvol` (Kastner and Hosszejni, 2019; Hosszejni and Kastner, 2019).

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Let $\Theta = (\mu, \beta, \phi, \sigma)$ represent the parameters, denote the observations by $y_{1:n} = \{y_1, \dots, y_n\}$, and the states (log-volatility) by $x_{0:n} = \{x_0, x_1, \dots, x_n\}$, with x_0 being the initial state. To run a full Gibbs sampler, we alternate between sampling model parameters and latent state sequences from their respective full conditional distributions. Letting $p(\cdot)$ denote a generic density, we have the following:

Procedure 1 (Generic Gibbs Sampler for State Space Models).

- (i) Draw $\Theta' \sim p(\Theta \mid x_{0:n}, y_{1:n})$.
- (ii) Draw $x'_{0:n} \sim p(x_{0:n} \mid \Theta', y_{1:n})$.

Procedure 1-(i) is generally much easier because it conditions on the complete data $\{x_{0:n}, y_{1:n}\}$. Procedure 1-(ii) amounts to sampling from the joint smoothing distribution of the latent state sequence and is generally more difficult. For linear Gaussian models, however, both parts of Procedure 1 are relatively easy to perform (Frühwirth-Schnatter, 1994; Carter and Kohn, 1994; Shumway and Stoffer, 2017, Chap. 6).

Although the focus of this note is to address Procedure 1-(i), we mention our preferred method for handling Procedure 1-(ii). Andrieu *et al.* (2010) introduced the particle Markov chain Monte Carlo (PMCMC) method, which proposed a conditional particle filter (CPF). The brilliance of the approach is that the CPF is invariant in the sense that the kernel leaves $p(x_{0:n} \mid \Theta', y_{1:n})$ invariant; that is, all elements of the chain have the target distribution. Thus, one can avoid methods that merely approximate the distribution.

Unfortunately, CPF suffers from the path degeneracy. Lindsten *et al.* (2014) solved the problem by introducing CPF with ancestral sampling (CPF-AS). The addition of ancestral sampling improved on the problem of path degeneracy while being robust to the number of particles generated. In fact, the method works very well with a small number of particles. Ancestral sampling maintains the invariance of the kernel and, in addition to being efficient, the CPF-AS algorithm is uniformly ergodic under rather general assumptions (Lindsten *et al.*, 2015). The algorithm is presented in the next section so as not to breakup the exposition.

As previously stated, step Procedure 1-(i) is typically the easier step. For state space models, normal priors on μ or β and ϕ [or a beta prior on $\frac{1}{2}(\phi + 1)$], and an inverse gamma prior on σ^2 are typically employed. The methods treat ϕ as a regression parameter and σ as scale parameter and this treatment is what leads to the inefficiency for this particular nonlinear model. The problem for SV models is that ϕ behaves more like a scale parameter than a regression parameter. For example, the autocorrelation function (ACF) of $\{y_t^2\}$ is given by

$$\text{Cor}(y_t^2, y_{t+h}^2) = \frac{\exp(\sigma_x^2 \phi^h) - 1}{\kappa_e \exp(\sigma_x^2) - 1}, \quad h = 1, 2, \dots, \quad (4)$$

where κ_e is the kurtosis of the noise, ϵ_t , and $\sigma_x^2 = \sigma^2 / (1 - \phi^2)$. For SV models, the ACF values are small and the decay rate as a function of lag is less than exponential and somewhat linear. This means that if you specify values for ϕ but allow us to control σ (and consequently σ_x), we can make the model ACF to look approximately the same no matter which values of ϕ are chosen. This is accomplished by moving ϕ and σ in opposite directions. Another way of looking at the problem is to define (with $\mu = 0$ and $\beta = 1$) $\xi_t = \frac{1}{2\sigma_x} x_t$ and $\zeta_t = \frac{1}{2} w_t$ so we may write (3) as

$$y_t = e^{\sigma_x \phi \xi_{t-1}} e^{\sigma \zeta_t} \epsilon_t, \quad (5)$$

noting that ξ_{t-1} and ζ_t are independent stationary $\frac{1}{2} N(0, 1)$ s. It is clear from (5) that σ and ϕ are scale parameters of the ξ_t process and σ is a scale parameter of the ζ_t noise process; we see that we can keep the scale of the data approximately the same by moving ϕ and σ in opposite directions.

For example, Figure 1 shows two data sequences (A and B) of length 1000 generated from two different SV models, (2)–(3), with $(\mu = 0, \beta = 1)$ MODEL I: $\phi = .99, \sigma = .15$, and MODEL II: $\phi = .95, \sigma = .35$, both with normal innovations. The figure also compares the sample ACF of each generated series squared (A and B) and

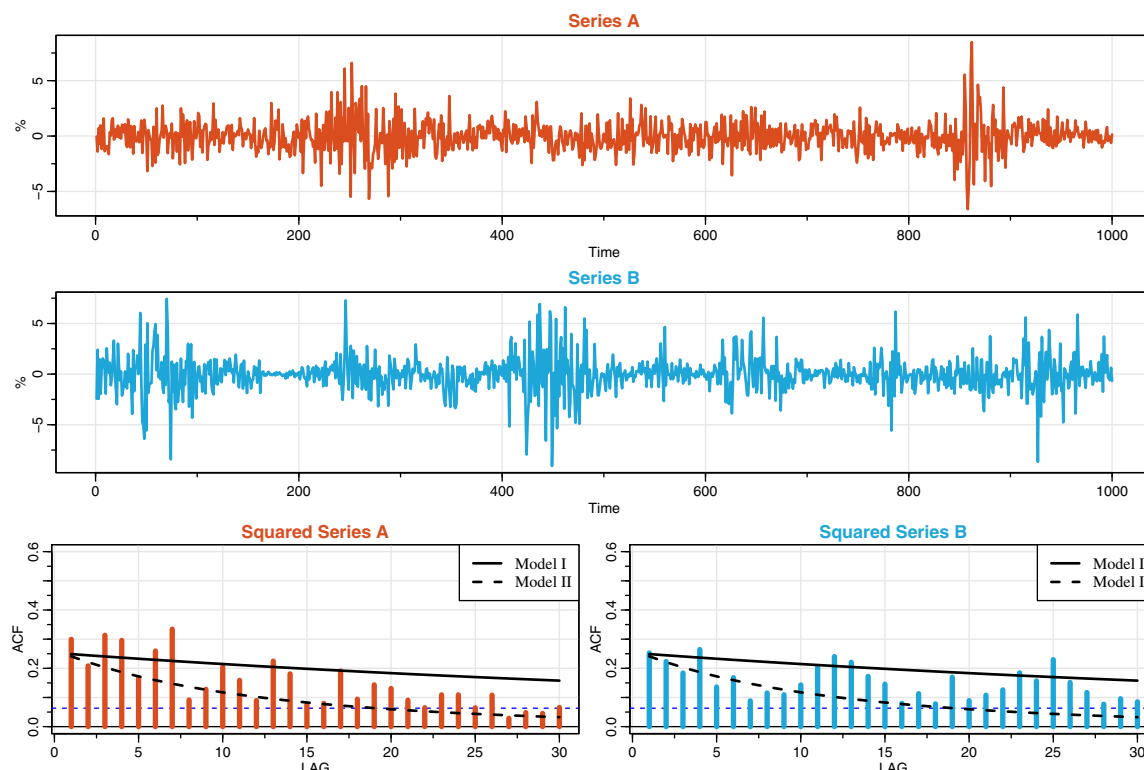


Figure 1. TOP: Two data sequences (A and B) of length 1000 generated from different two-parameter SV models, I: $\phi = .99$, $\sigma = .15$ and II: $\phi = .95$, $\sigma = .35$. BOTTOM: The ACF of each generated series squared (A and B) and the theoretical ACFs of SV models I and II as lines. Series A corresponds to Model I and Series B corresponds to Model II

the theoretical counterparts for Models I and II. While the AR parameter, ϕ , is very different in each model, the simulated series look very much the same. In addition, the sample ACFs suggest that Series B is from Model I when in fact it is from Model II.

While CPF-AS can improve the mixing of the sampler for Procedure 1-(ii), it does not address mixing for Procedure 1-(i). The problem persists to this day as one can see by using the R package `stochvol` (Kastner and Hosszejni, 2019). While there are various methods that jointly sample ϕ and σ^2 (e.g., Kim *et al.*, 1998; Kastner and Frühwirth-Schnatter, 2014) the samplers do not improve the efficiency because the methods do not exploit the geometry of the target, which we will address in due course.

As an example of the state of the art, we took an example out of one of the vignettes from the `stochvol` package previously mentioned (Kastner, 2016). The example used in the vignette is to fit an SV model to the returns of the daily EUR/CHF exchange rates (the example uses t observational errors). The code we use is taken directly from the example, and consequently, our display matches the one given in the vignette. Figure 2 shows the results of the analysis. The top row shows the draws of ϕ and σ in order, the bottom left shows the sample ACF of the traces, and on the right, there is a scatterplot of the pairs of values in each draw. The poor mixing and inefficiencies in this example are evident, and it is clear that the slow convergence is due to the *chattering* (a type of meandering) through the sample spaces of the parameters.

The effective sample size (ESS), which measures the amount by which the dependence structure of the sampling routine increases uncertainty in the estimates, was estimated for this example using the `mcmc` R package (Geyer and Johnson, 2017) to be $ESS = 276$ for the ϕ trace, and $ESS = 109$ for the σ trace. This means that, although the estimated posteriors would be based on 10,000 samples, the essential sample sizes are less than 3% of the number

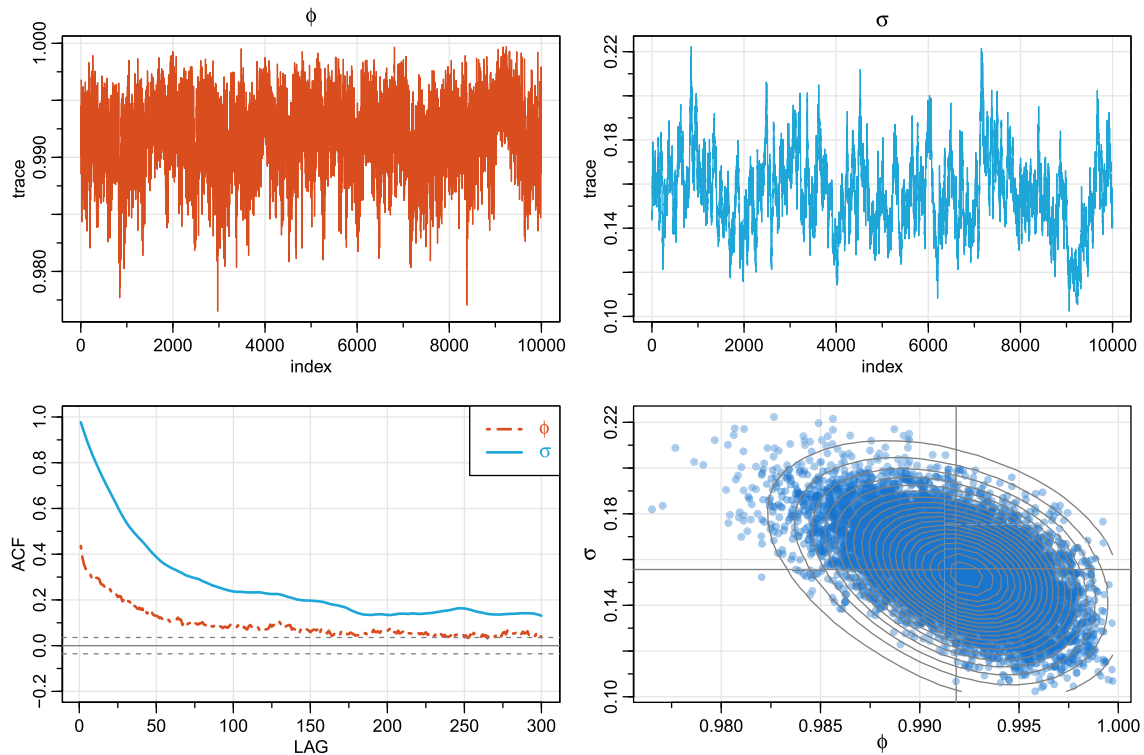


Figure 2. The sampling results for ϕ and σ taken directly from an example from the `stochvol` R package analyzing the daily EUR/CHF exchange rates. TOP: Separate draws of ϕ and σ in order of the 10,000 draws after a burn-in of 2000. BOTTOM: The sample ACF of the traces (left), and a scatterplot (right) of the pairs of values in each draw with the posterior means highlighted. The ESSs in this case are estimated to be 276 for the ϕ trace and 109 for the σ trace

of draws. The estimated ESS is based on the estimate of inefficiency (IF) as defined in Geyer (1992),

$$\text{IF} := 1 + 2 \sum_{i=1}^{\infty} \rho(i), \quad (6)$$

where $\rho(i)$ is the autocorrelation function of the trace at lag i . In particular, the estimated ESS is given by the number of draws divided by the estimate of IF. Random sampling would yield an ESS equal to the number of draws (or $\text{IF} = 1$) and it is possible to do better than random sampling; details may be found in Shumway and Stoffer (2017, §1.5).

To improve on efficiency, we propose a new method for SV models by employing a particular bivariate prior and sampling the parameters jointly from $p(\phi, \sigma \mid x_{0:n}, y_{1:n})$. The new method reduces the sampling inefficiencies significantly because it exploits the geometry of the target distribution. A random walk Metropolis algorithm is used to implement the method. In all of our examples, our MCMC routines use relatively small sample and burn-in sizes because the gain in efficiency from the proposed method is significant.

2. CONDITIONAL PARTICLE FILTERING WITH ANCESTRAL SAMPLING

For the sake of completeness and before we discuss our method, we present the CPF-AS algorithm that we use to perform step Procedure 1-(ii). The goal is to repeatedly draw an entire state sequence from the posterior $p(x_{0:n} \mid$

$\Theta, y_{1:n}$). To ease the notation, we will drop the conditioning arguments in this section. Many of the details (along with references) may be found in Lindsten *et al.* (2014) and Douc *et al.* (2014, Part III).

For notation, we will denote the proposal density by $q(\cdot)$, the target density by $p(\cdot)$, and the importance function (un-normalized weight) by $\omega = p/q$. Every density shown is conditional on parameters Θ and data $y_{1:t}$ up to time t . At the end of the procedure, we will have a sample of size N from the target of interest, $p(x_{0:n} | \Theta, y_{1:n})$. To keep the exposition simple, a resampling step that was described in Gordon *et al.* (1993) and subsequently improved by others, and an auxiliary adjustment step as described in Pitt and Shephard (1999) are applied appropriately in the procedures, but we do not explicitly show these steps.

Procedure 2 (Conditional Particle Filter with Ancestral Sampling – [CPF-AS]).

INPUT: A sequence of conditioned particles $x'_{0:n}$ as a reference trajectory.

- (i) Initialize, $t = 0$:
 - (a) Draw $x_0^j \sim q(\cdot)$ for $j = 1, \dots, N-1$ (sample only $N-1$ particles).
 - (b) Set the N th particle, $x_0^N = x'_0$.
 - (c) Compute weights $\omega_0^j \propto \omega_0(x_0^j)$ for $j = 1, \dots, N$.
- (ii) for $t = 1, \dots, n$:
 - (a) Draw $I_t^j \sim \text{Discrete}(\{\omega_{t-1}^i\}_{i=1}^N)$ for $j = 1, \dots, N-1$.
 - (b) Draw $x_t^j \sim q(x_t | x_{0:t-1}^{I_t^j})$ for $j = 1, \dots, N-1$.
 - (c) Set $x_t^N = x'_t$.
 - (d) Draw $I_t^N \sim \text{Discrete}(\{\omega_{t-1}^i\}_{i=1}^N)$ (ancestor sample).
 - (e) Set $x_{0:t}^j = (x_{0:t-1}^{I_t^j}, x_t^j)$ and $\omega_t^j \propto \omega_t(x_{0:t}^j)$, for $j = 1, \dots, N$.

While other methods exist, we note again that the resulting Markov kernel leaves its target distribution, $p(x_{0:n} | \Theta, y_{1:n})$, invariant, regardless of the number of particles (Lindsten *et al.*, 2014) and under general conditions is uniformly ergodic (Lindsten *et al.*, 2015). Hence, Procedure 2 enables fast mixing of the particle Gibbs kernel even when using a few particles.

3. PROPOSED METHOD

In the SV model, (2)–(3), β and μ are not both needed. In choosing which parameter to keep, Kim *et al.* (1998) argued that allowing μ to vary and fixing $\beta = \exp(\mu/2)$ has a better interpretation from an economic point-of-view. In this section, we will follow their restriction on β and allow μ to vary. It has been our experience, however, that from a statistical point-of-view, it is better to keep β and not use μ . Because β is a scale parameter of the observational innovation sequence, its presence allows forming various wide tail distributions such as the t -distribution or the Laplacian or double exponential distribution as described in Andrews and Mallows (1974). We explore this case further in Section 5.

In Section 1, we discussed the problems of applying MCMC methods to SV models. Although various techniques such as CPF-AS exist that help solve some of the slow convergence problems when performing Procedure 1-(ii), we still observe poor mixing in Procedure 1-(i) caused by the inverse dependence between ϕ and σ . In this section, we suggest a method to improve the convergence by exploiting the geometry of the target.

To accomplish this goal, we put a bivariate normal prior with a negative correlation coefficient on the pair $\Theta = (\phi, \sigma)$,

$$\begin{pmatrix} \phi \\ \sigma \end{pmatrix} \sim N_2 \left(\begin{bmatrix} \mu_\phi \\ \mu_\sigma \end{bmatrix}, \begin{bmatrix} \sigma_\phi^2 & \rho\sigma_\phi\sigma_\sigma \\ \rho\sigma_\phi\sigma_\sigma & \sigma_\sigma^2 \end{bmatrix} \right), \quad (7)$$

where $\rho < 0$. Allowing possible negative values for σ is an old trick used in optimization to avoid constraints on the parameter space and is akin to the use of the Cholesky decomposition when estimating a covariance matrix to ensure the non-negative definiteness of the matrix. In this case, the draws corresponding to σ^2 will always be non-negative and marginally, has a scaled chi-squared prior distribution. In addition, as is seen in Figure 2, a bivariate normal prior is sensible because it appears that joint distribution of (ϕ, σ) is elliptically symmetric. This fact is evident from other examples as will be seen. The proposition of sampling from (7) is in line with the consideration that exploiting the geometry of the target can improve the efficiency of an MCMC algorithm (Robert *et al.*, 2018).

Note that we have changed the notation slightly by excluding μ from $\Theta = (\phi, \sigma)$ because it may be sampled separately if necessary. To accomplish Procedure 1-(i), note that,

$$p(\Theta, \mu \mid x_{0:n}, y_{1:n}) \propto \pi(\Theta, \mu) p(x_0 \mid \Theta, \mu) \prod_{t=1}^n p(x_t \mid x_{t-1}, \Theta, \mu) p(y_t \mid x_t, \Theta, \mu), \quad (8)$$

where $\pi(\Theta, \mu)$ is the prior on the parameters. For the generic state space model, the parameters are often taken to be conditionally independent with distributions from standard parametric families (at least as long as the prior distribution is conjugate relative to the model specification). In this case, however, we must work with non-conjugate models, and one option is to replace Procedure 1-(i) with a Metropolis–Hastings step, which is feasible because the complete data density $p(\Theta, \mu, x_{0:n}, y_{1:n})$ can be evaluated pointwise.

Under these considerations, for the SV model in (2)–(3), we have

$$\begin{aligned} p(\Theta \mid \mu, x_{0:n}, y_{1:n}) &\propto \pi(\Theta) p(x_0 \mid \Theta, \mu) \prod_{t=1}^n p(x_t \mid x_{t-1}, \Theta, \mu) \\ &\propto \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(\phi - \mu_\phi)^2}{\sigma_\phi^2} + \frac{(\sigma - \mu_\sigma)^2}{\sigma_\sigma^2} - \frac{2\rho(\phi - \mu_\phi)(\sigma - \mu_\sigma)}{\sigma_\phi \sigma_\sigma} \right] \right\} \\ &\quad \cdot \frac{\sqrt{1-\phi^2}}{\sigma} \exp \left\{ -\frac{(x_0 - \mu)^2}{2\sigma^2/(1-\phi^2)} \right\} \prod_{t=1}^n \frac{1}{\sigma} \exp \left\{ -\frac{[(x_t - \mu) - \phi(x_{t-1} - \mu)]^2}{2\sigma^2} \right\} \\ &\propto \exp \left\{ -\frac{(\phi - \mu_\phi)^2 \sigma_\sigma^2 + (\sigma - \mu_\sigma)^2 \sigma_\phi^2 - 2\rho \sigma_\phi \sigma_\sigma (\phi - \mu_\phi)(\sigma - \mu_\sigma)}{2(1-\rho^2) \sigma_\phi^2 \sigma_\sigma^2} \right\} \\ &\quad \cdot \frac{\sqrt{1-\phi^2}}{\sigma^n} \exp \left\{ -\frac{(1-\phi^2)(x_0 - \mu)^2 + \sum_{t=1}^n [(x_t - \mu) - \phi(x_{t-1} - \mu)]^2}{2\sigma^2} \right\}. \end{aligned} \quad (9)$$

As previously suggested, we use a random walk Metropolis step to sample $\Theta = (\phi, \sigma)$ simultaneously from the target posterior distribution $p(\Theta \mid \mu, x_{0:n}, y_{1:n})$ given in (9). This approach involves choosing a tuning parameter to control the acceptance probability. However, sometimes a good proposal distribution is difficult to choose because both the size and the spatial orientation of the proposal distribution should be considered. We have found that occasionally, the use of an adaptive method can help with the problem and we suggest using a technique that was presented in Andrieu and Thoms (2008, Alg. 4).

Procedure 3 (Sampling Θ).

INPUT: An initial value, Θ_0 , and an initial bivariate normal proposal distribution $N_2(\mu_0, \lambda_0 \Sigma_0)$.

On iteration $j + 1$, for $j = 0, 1, 2, \dots$,

- (i) Draw $\vartheta \sim N_2(\Theta_j, \lambda_j \Sigma_j)$ and set $\Theta_{j+1} = \vartheta$ with probability $\alpha_{j+1} = \frac{g(\vartheta)}{g(\Theta_j)} \wedge 1$, where $g(\Theta)$ is given on the RHS of (9). Otherwise, set $\Theta_{j+1} = \Theta_j$.

Algorithm 1 Joint particle Gibbs for stochastic volatility models

INPUT: Set the initial value of $\Theta^{[0]} = (\phi, \sigma)^{[0]}$, $\mu^{[0]}$, and $x_{0:n}^{[0]}$ arbitrarily.

At iteration $j = 1, 2, \dots$,

- (i) Draw $x_{0:n}^{[j]}$ by CPF-AS, Procedure 2, conditioned on $x_{0:n}^{[j-1]}$ and $\Theta^{[j-1]}, \mu^{[j-1]}$.
 - (ii) With $x_{0:n}^{[j]}$, generate $\Theta^{[j]} = (\phi, \sigma)^{[j]}$ via Procedure 3 and draw $\mu^{[j]}$ from the posterior given in (13) under the current draws $x_{0:n}^{[j]}$ and $\Theta^{[j]}$.
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(ii) [Optional Adaptive Method] Update

$$\log(\lambda_{j+1}) = \log(\lambda_j) + \gamma_{j+1}[\alpha_{j+1} - \alpha_*], \quad (10)$$

$$\mu_{j+1} = \mu_j + \gamma_{j+1}(\Theta_{j+1} - \mu_j), \quad (11)$$

$$\Sigma_{j+1} = \Sigma_j + \gamma_{j+1}[(\Theta_{j+1} - \mu_j)(\Theta_{j+1} - \mu_j)' - \Sigma_j], \quad (12)$$

where γ_j is a scalar non-increasing sequence of positive step lengths such that $\sum_{j=1}^{\infty} \gamma_j = \infty$ and $\sum_{j=1}^{\infty} \gamma_j^{1+\delta} < \infty$ for some $\delta > 0$; α_* is the expected acceptance rate for the algorithm.

If Step (ii) is skipped, keep the tuning parameter and the covariance matrix fixed at λ_0 and Σ_0 respectively.

The optional part makes the algorithm non-Markovian; however, it can adapt continuously to the target distribution. Both the size and the spatial orientation of the proposal distribution will be adjusted by the adaptation procedure. Also, Procedure 3 is straightforward to implement and to use in practice. There are no extra computational costs because only a simple recursion formula for the covariances involved. The algorithm starts by using the accumulating information from the beginning of the sampling and it ensures that the search becomes more efficient at an early stage of the sampling. Haario *et al.* (2001) establish that the adaptive MCMC algorithms do indeed have the correct ergodicity properties. The hyperparameters in (7) should be chosen with care to ensure an optimal acceptance rate of about 28% (see Gelman *et al.*, 1996) and we suggest that ρ be moderately negative, $\rho \in (-.6, -.3)$. In addition, if adaptation is not used, the tuning parameter λ_0 should be chosen to maintain a proper acceptance rate; see Douc *et al.* (2014, Example 5.37) for more details.

If the parameter μ is included in the model, using a diffuse prior (e.g., see Kim *et al.*, 1998), we have

$$\mu \mid \Theta, x_{0:n}, y_{1:n} \sim N(\nu_\mu, \sigma_\mu^2) \quad (13)$$

where

$$\nu_\mu = \sigma_\mu^2 \left\{ \frac{1 - \phi^2}{\sigma^2} x_0 + \frac{1 - \phi}{\sigma^2} \sum_{t=1}^n (x_t - \phi x_{t-1}) \right\}$$

and

$$\sigma_\mu^2 = \frac{\sigma^2}{n(1 - \phi)^2 + (1 - \phi^2)}.$$

Recall that we are fixing $\beta = \exp(\mu/2)$. Finally, our algorithm for the analysis of a SV model is given in Algorithm 1.

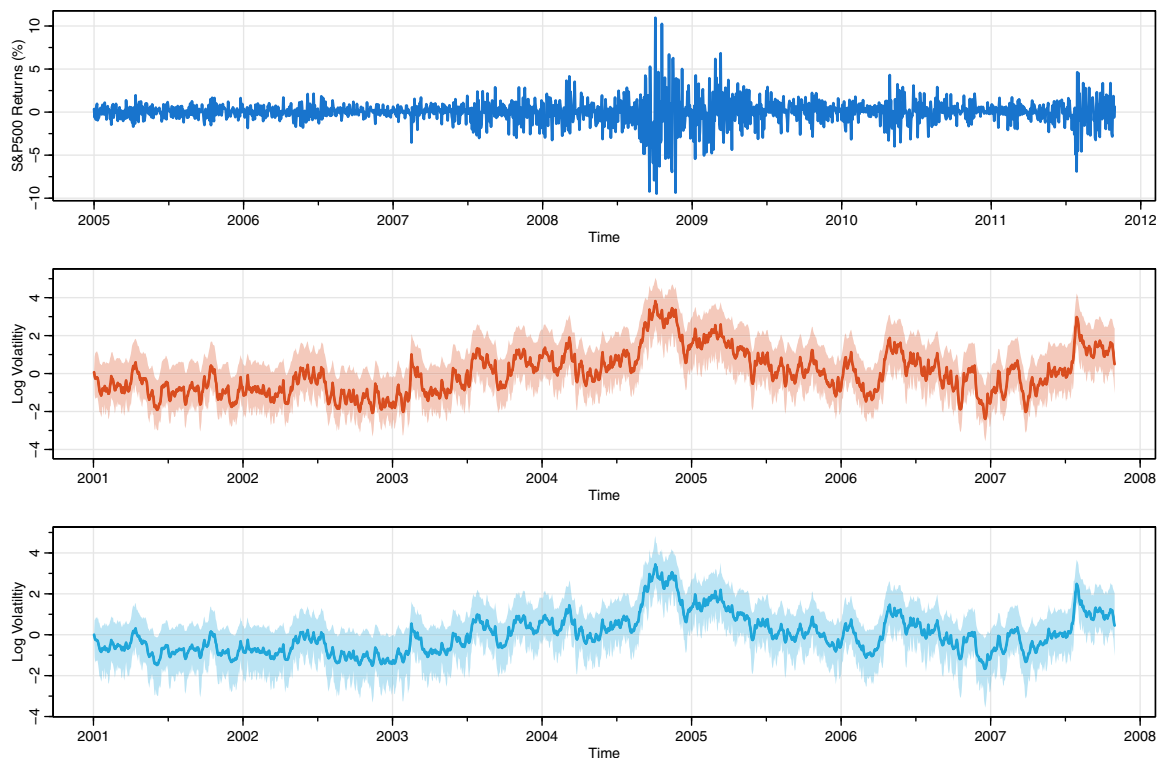


Figure 3. TOP: Daily returns of the S&P 500 from January 2005 to October of 2011. MIDDLE: State process estimated posterior mean and pointwise 95% credible intervals based on a slow mixing sampler. BOTTOM: State process estimated posterior mean and pointwise 95% credible intervals based on the proposed method of structured joint parameter sampling, Algorithm 1. The number of particles used in the particle filter (Procedure 2) for both methods was $N = 20$

4. EXAMPLES

4.1. Two-Parameter Model

In this section, we fit a two-parameter model ($\mu = 0$) to the daily returns of the S&P 500 from January 2005 to October of 2011 shown at the top of Figure 3. The data include the financial crisis of 2008. We compare two methods using CPF-AS (Procedure 2) to sample the state process in both. The first method samples the parameters individually by drawing from the univariate conditionals $p(\phi \mid \sigma, x_{0:n}, y_{1:n})$ and $p(\sigma \mid \phi, x_{0:n}, y_{1:n})$ using the typical normal-inverse gamma conjugate prior for an AR model, while our method exploits the geometry of the target and samples the parameters jointly as described in Algorithm 1 holding μ at zero.

In each case, we used $N = 20$ particles for the CPF-AS (Procedure 2) and 5000 iterations after a burn-in of 100. The posterior mean and a pointwise 95% credible interval of the draws of the state (log-volatility) process are shown in Figure 3. The middle plot shows the results for the typical method and the bottom plot shows the results for the proposed method. The results are similar, but the trace of the estimated process via the proposed method is smoother and less variable than the individual method shown in the middle.

Figure 4 displays the results of the parameter estimation using the one-at-a-time method of sampling ϕ and σ . The top of the figure shows the traces of the sampled values after burn-in. The corresponding posterior means are about .88 for ϕ and .62 for σ . The bottom of the figure shows the sample ACFs of the traces and a scatterplot of the sampled values. The estimated inefficiencies are displayed with the sample ACFs of traces. We note again the slow convergence problem seen in the simulation example. Finally, the bottom right scatterplot indicates that the

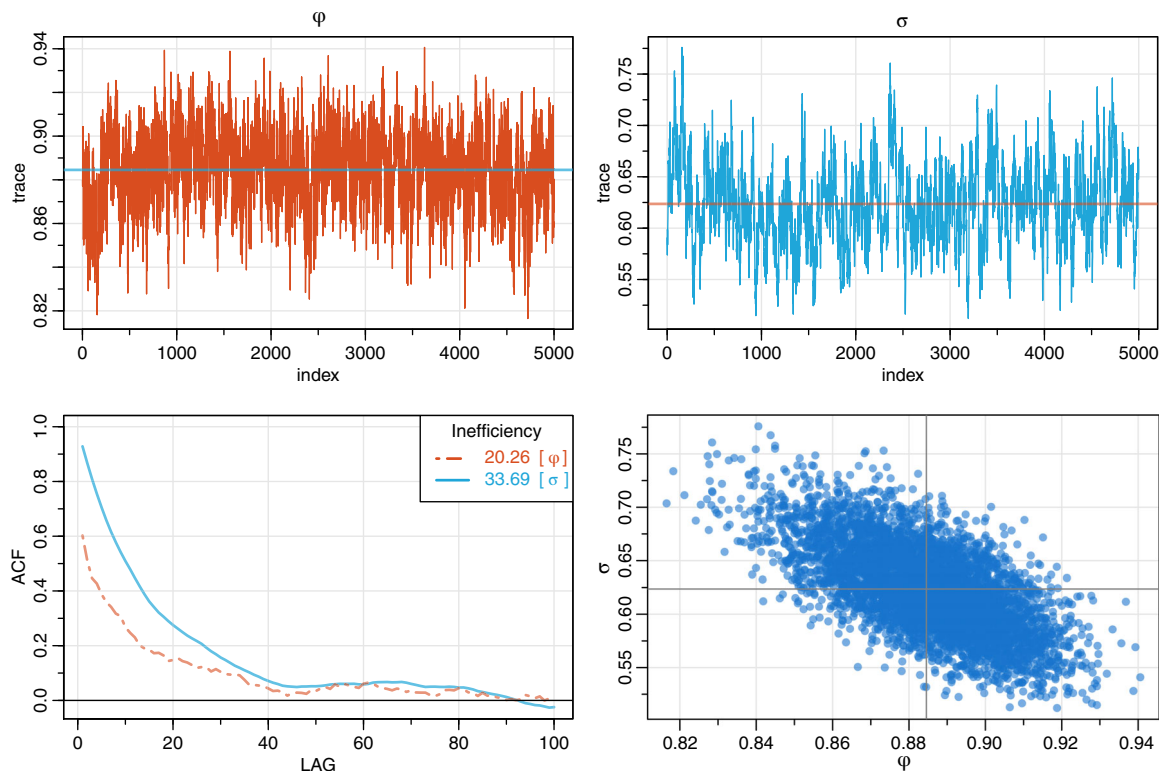


Figure 4. **(Individual Sampling)** S&P 500 results of the parameter estimation in a two-parameter model when ϕ and σ are sampled separately. The top shows the traces of the 5000 sampled values after a burn-in of 100. The number of particles used in the particle filter (Procedure 2) was $N = 20$. The corresponding posterior means are .88 for ϕ and .62 for σ . The bottom left shows the sample ACFs of the traces and the estimated inefficiency measure as defined in (6). The bottom left shows a scatterplot of the sampled parameters, and exhibits strong negative correlation of $-.61$

joint posterior of ϕ and σ is elliptical and symmetric with negative orientation. We saw this behavior in Figure 2, and it seems to be common.

Figure 5 displays the results of the parameter estimation using our structured method, Algorithm 1, jointly sampling ϕ and σ from an elliptically symmetric distribution (specifically, bivariate normal) with negative orientation. The top of the figure shows the traces of the sampled values after burn-in. The corresponding posterior means are .80 for ϕ and .36 for σ . The bottom of the figure shows the sample ACFs of the traces and a scatterplot of the sampled values, which shows an improvement of the established method. In fact, the inefficiencies in the typical sampling method are more than twice a big than in the proposed method.

4.2. Three-Parameter Model

Next, we fit a three-parameter SV model to the S&P 500 series using Algorithm 1; the adaptive part of the Metropolis step, Procedure 3-(ii), was not needed here. To keep the complexity low, we used only $N = 10$ particles for sampling the states (Procedure 2), and then generated 2000 samples after a burn-in of 100. The acceptance rate was nearly optimal at 26.1%.

The results of the parameter estimation are shown in Figure 6; the results for the state estimation are similar to the lower plot of Figure 3 and are not shown to save space. The figure shows the trace of the draws (top row), the sample ACF of the draws (middle row) along with the estimated inefficiency, (6), and a histogram of the results

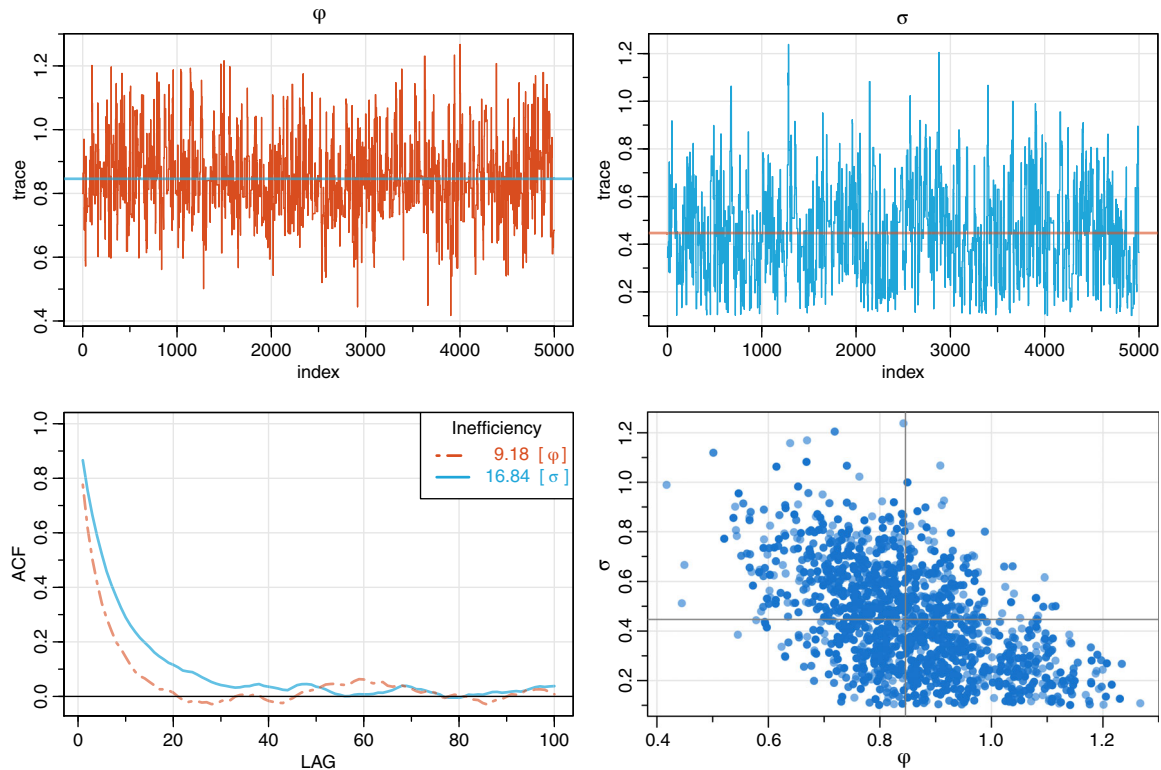


Figure 5. **(Joint Sampling)** S&P 500 results of the parameter estimation in a two-parameter model using the proposed method of sampling ϕ and σ jointly, Algorithm 1. The top shows the traces of the 5000 sampled values after a burn-in of 100. The number of particles used in the particle filter (Procedure 2) was $N = 20$. The corresponding posterior means are .85 for ϕ and .45 for σ and the correlation between the draws is $-.55$. The bottom shows the sample ACFs of the traces and a scatterplot of the sampled values, which indicate the sample parameters are uncorrelated. In addition, estimated inefficiency measures are improved over the counterparts in Figure 4

(bottom row). The posterior means are displayed in the figure and were .85 for ϕ , .43 for σ and $-.01$ for μ . We note that the results are quite good even using this quick analysis. In fact, the efficiency for estimating μ in this case is better than independent sampling ($IF = .76$). It is also apparent that the previous analysis based on the two-parameter model ($\mu = 0$) was reasonable.

5. STOCHASTIC VOLATILITY WITH MULTIPLE OBSERVATIONS

It is often reasonable to assume that similar assets are being driven by the same volatility process. In this case, the model presented in Asai *et al.* (2006) can be used. The model assumes a univariate volatility process is driving a number of similar assets and is given by,

$$x_t = \phi x_{t-1} + \sigma w_t \quad (14)$$

$$y_{it} = \beta_i \exp\left\{\frac{x_t}{2}\right\} \epsilon_{it}, \quad i = 1, \dots, p, \quad (15)$$

where the y_{it} are the returns of the i th asset, $w_t \stackrel{\text{iid}}{\sim} N(0, 1)$, and $\epsilon_t = (\epsilon_{1t}, \dots, \epsilon_{pt})' \stackrel{\text{iid}}{\sim} N_p(0, I)$. In this model, μ is removed to avoid overparameterization and each β_i is a scale parameter for the i th asset.

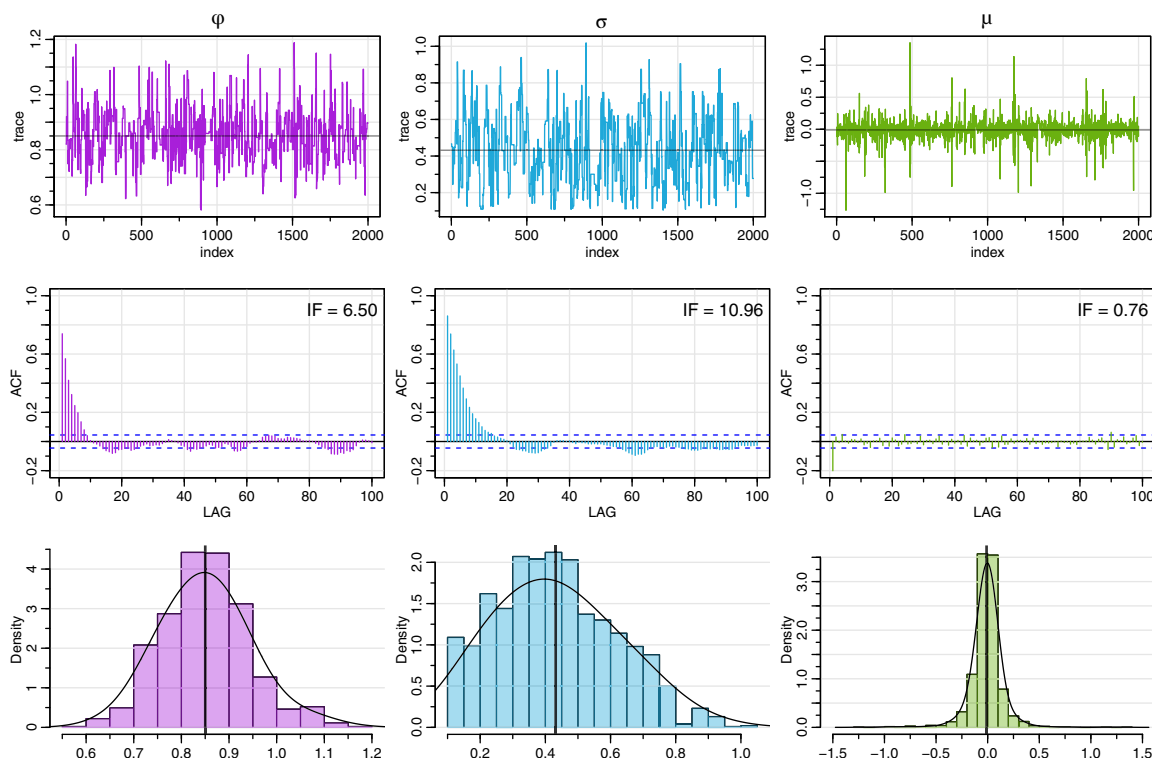


Figure 6. S&P 500, results of the parameter estimation in a three-parameter model using the proposed method of sampling ϕ and σ jointly, Algorithm 1. TOP: Traces of the 2000 sampled values of the parameters after burn-in. MIDDLE: Sample ACF of the draws along with the estimated inefficiency, (6). BOTTOM: Histogram of the results. The posterior means are displayed in the figure and were .85 for ϕ , .43 for σ and $-.01$ for μ . The correlation between the joint draws of ϕ and σ is $-.54$

We can easily apply our proposed method to this model. That is, as in the univariate case, we put a bivariate normal prior on the state parameters, ϕ and σ . Then, because β_i , for $i = 1, \dots, p$, is a scale parameter, a reasonable choice is to use independent inverse gamma priors for β_i^2 . That is, if $\beta_i^2 \sim \text{IG}(a_i/2, b_i/2)$, then the posterior is

$$\beta_i^2 \mid \Theta, x_{0:n}, y_{1:n} \sim \text{IG}\left(\frac{1}{2}(a_i + n + 1), \frac{1}{2}\left\{b_i + \sum_{t=1}^n \frac{y_{it}^2}{\exp(x_t)}\right\}\right). \quad (16)$$

Hence, for this model, we can simply add a third step to Algorithm 1, which is to sample β_i^2 from (16) for $i = 1, \dots, p$. In addition, the use of an inverse gamma prior on the β_i^2 implies the marginal distribution of each observational innovation is a t -distribution; that is, $e_{it} \mid \beta_i \sim \text{N}(0, \beta_i^2)$ and $\beta_i^2 \sim \text{IG}(a_i/2, b_i/2)$ implies e_{it} has a t -distribution with location 0, shape a_i , and scale b_i/a_i as discussed in Andrews and Mallows (1974). We summarize the steps in Algorithm 2.

For an example, we consider the daily NYSE returns for three banks, Bank of American (BOA), Citigroup (Citi), and J.P. Morgan (JPM) from January 2005 to November 2017. The data are displayed in Figure 7; also shown is the estimated log-volatility, which we describe shortly.

We used Algorithm 2 with $N = 20$ particles to generate 2000 draws after a burn-in of 200 iterations. The procedure was non-adaptive and the acceptance rate was 29.8%. The parameter estimation summary is displayed in Figure 8 and the display is similar to the previous examples. The displays suggest that the algorithm is mixing well. The top shows the traces of the draws for each parameter and indicates the posterior means, .86 for ϕ , .32

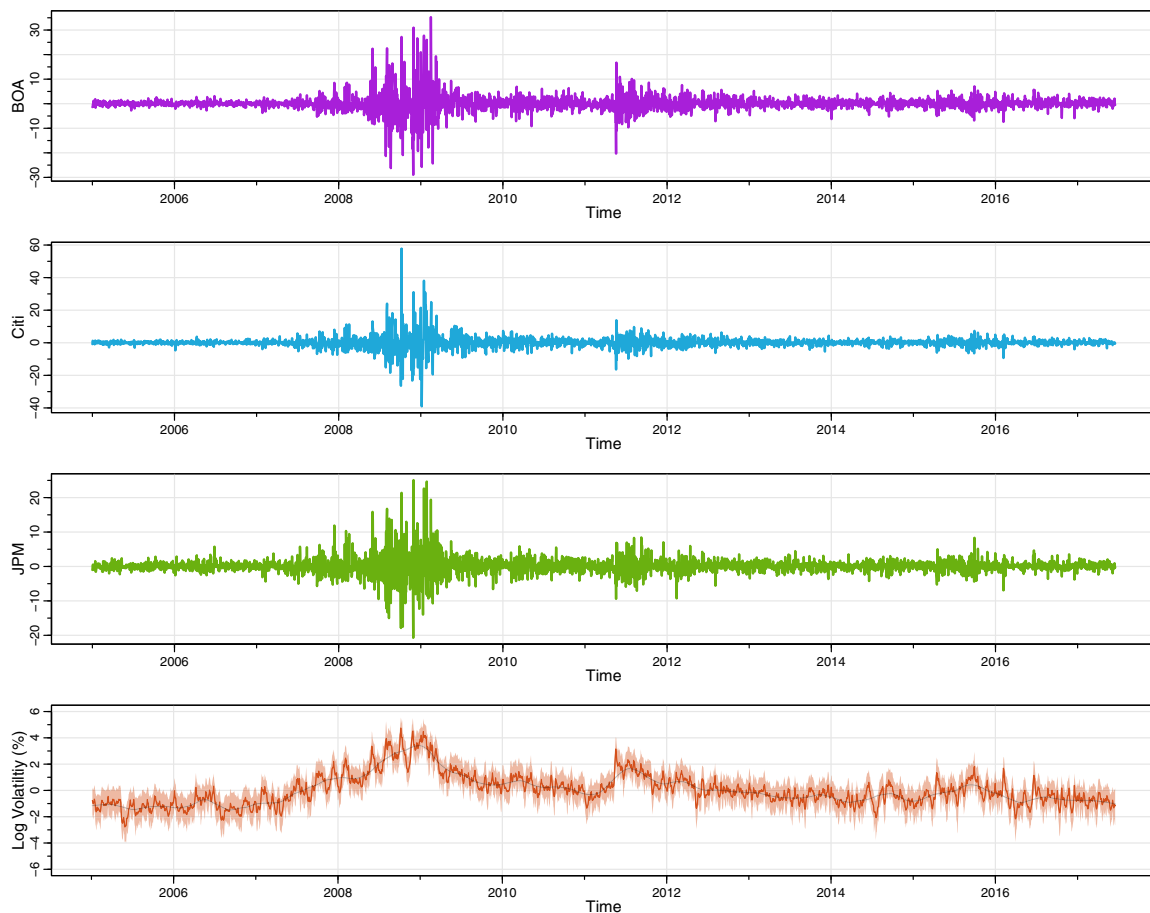


Figure 7. The daily NYSE returns (as percentages) for three banks, Bank of American (BOA), Citigroup (Citi), and J.P. Morgan (JPM) from January 2005 to November 2017. BOTTOM: The resulting posterior of the log-volatility based on Algorithm 2. Shown are the posterior mean and a swath displaying the limits of 99% of the sampled states. We also display a lowess smooth as a thin line to emphasize the volatility trend

for σ , and 1.64, 1.62, and 1.42 for the β s of BOA, Citi, and JPM respectively. The middle plot shows the sample ACFs of the traces along with the inefficiencies, which are acceptable. The bottom row of Figure 8 displays the posterior distributions of each parameter along with the location of the posterior mean.

The resulting posterior of the log-volatility is shown at the bottom of Figure 7. Shown are the posterior mean and a swath displaying pointwise 99% credible intervals. We also display a lowess fit as a thin line to emphasize the volatility trend. Notice that the impending financial crisis of 2008 is visible at least one year prior as the volatility starts a trend upwards just prior to 2007. It seems that there is an advantage to using multiple similar sources to estimate volatility.

6. CLOSING REMARKS

The conditional particle filter with ancestral sampling (Procedure 2) was a breakthrough for analyzing nonlinear state space models by establishing a computationally efficient, invariant and uniformly ergodic method of sampling from the posterior of the hidden state trajectories, Procedure 1-(ii). The method works well for many cases if drawing from the posterior of the parameters, Procedure 1-(i), is not problematic. Unfortunately, this situation does

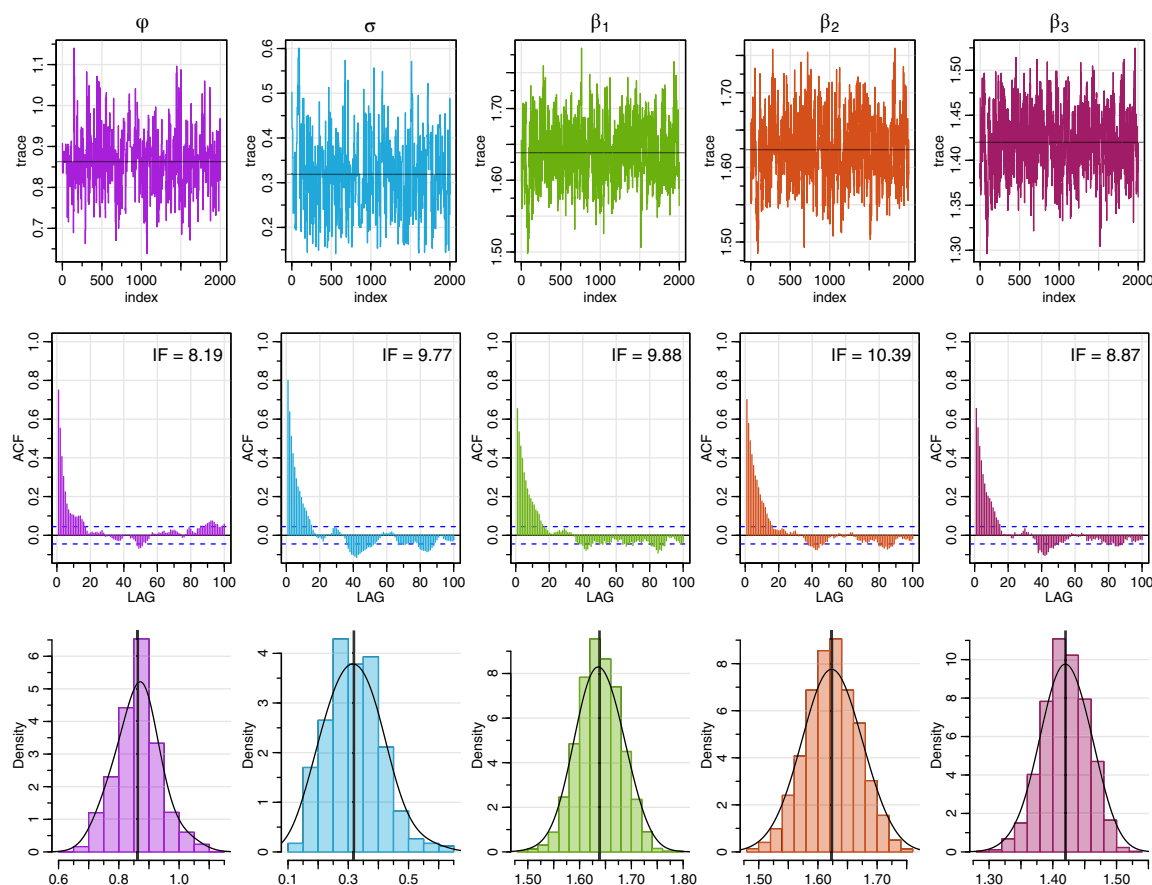


Figure 8. For the bank returns data shown in Figure 7, the results of fitting an MSV model based on Algorithm 2. TOP: The traces of the draws for each parameter and indicates the posterior means, .86 for ϕ , .32 for σ , and 1.64, 1.62, and 1.42 for the β s of BOA, Citi, and JPM respectively. MIDDLE: The sample ACFs of the traces along with the inefficiencies. BOTTOM: The estimated posterior distributions of each parameter along with the location of the posterior mean

not include the case of stochastic volatility models because in the state equation, the autoregressive parameter (ϕ) and the noise scale (σ) in (2) appear to have an elliptically symmetric distribution with negative orientation.

Prior attempts to handle SV models had less than optimal efficiency because ϕ was treated as a regression parameter while σ was treated as a scale parameter. Initially, these parameters were sampled individually as they typically are for state space models. Later, various researchers applied techniques that sampled ϕ and σ^2 jointly. These approaches, however, were problematic because the joint distributions do not exploit the elliptical geometry of the target.

We have presented a method that exploits the geometry of the target by sampling the state parameters jointly using a bivariate normal distribution. While it is possible that a sampled pair yields values of $|\phi| > 1$ or $\sigma < 0$, it is not a problem. For example, the state process is assumed to be stationary, so realistically, one only needs $|\phi| \neq 1$ (e.g., see Shumway and Stoffer, 2017, Example 3.4), which will not happen (with probability 1 in all but pathological cases). Also, sampled values of σ^2 will always be non-negative.

Although we did not discuss factor models, it seems clear that the method may be extended to the case where the state variable is a vector of independent volatility factors. In this case, (14) would be vector-valued with independent components and the joint distribution of the corresponding ϕ s and σ s would again be taken to be multivariate normal.

Algorithm 2 Joint particle Gibbs for (14)–(15)

INPUT: Set the initial value of $\Theta^{[0]} = (\phi, \sigma)^{[0]}$, $\beta_{1:p}^{[0]}$, and $x_{0:n}^{[0]}$ arbitrarily.

At iteration $j = 1, 2, \dots$,

- (i) Draw $x_{0:n}^{[j]}$ by CPF-AS, Procedure 2, conditioned on $x_{0:n}^{[j-1]}$ and $\Theta^{[j-1]}$.
 - (ii) With $x_{0:n}^{[j]}$, generate $\Theta^{[j]} = (\phi, \sigma)^{[j]}$ via Procedure 3 and draw $\beta_{1:p}^{2[j]}$ from the posteriors given in (16) under the current draws $x_{0:n}^{[j]}$ and $\Theta^{[j]}$.
-

7. DATA AVAILABILITY STATEMENT

We did not supply every particular numerical detail (e.g., hyperparameters and tuning parameters) of our examples in the text. Instead, for the sake of reproducibility, we supply the data and the R code (R Core Team, 2019) for every example on GitHub; see Gong and Stoffer (2020) for the URL. Additional information may also be found in Gong (2019). All of our examples were performed using Microsoft R Open (Microsoft and Team, 2019) version 3.5.3, which is the last version available. However, the example from the `stochvol` R package was done under vanilla R version 4.0.2.

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